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A LEVELSET METHOD IN SHAPE AND TOPOLOGY OPTIMIZATION FOR VARIATIONAL INEQUALITIES

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Abstract: Levelset method is used for shape optimization of the energy functional for Signorini problem. The topological derivatives are employed for the topology variations in the form of small holes. Numerical results confirm that the method is efficient and gives better results compared to the classical shape optimization techniques.

Keywords: Shape optimization, topological derivative, levelset method, variational inequality, asymptotic analysis

1 Introduction

In the present paper a numerical method for shape and topology optimization of the energy functional for Signorini problem is proposed. The method includes the shape gradients and topological derivatives of the functional in question, and the levelset method is used for the evolution of geometrical domains.

The Hamilton-Jacobi nonlinear hyperbolic equation models the evolution of the level set function. The normal speed of the moving boundaries are determined from the shape gradients obtained for the energy functional (see [25]).

The topology changes are defined by the topological derivatives of the energy functional. The small holes are injected into the actual geometrical domain with the centers at the points determined by maximization of the topological derivatives.

We provide also the arguments which allow us to determine the topological derivatives. To this end the domain decomposition technique is applied. The proof of the asymptotic expansion of Steklov-Poincaré operator used in such a technique is given in the appendix. The technique for such an analysis is proposed in [23] and [8].

Singular perturbations of domains in the framework of shape optimization are studied in [1], [4], [5], [9], [10], [14], [15], [11], [12], [13], [20], [21], [22]. The construction of the asymptotic expansion for the Steklov-Poincaré operator is given in [24].

2 The Signorini problem

We introduce the model problem. Let U and V be two bounded open subsets of $\mathbb{R}^2$ such that $V \subset\subset U$. For any open set $\omega \subset \mathbb{R}^2$, we denote by $\#\overline{\omega}$ the number of connex components of $\overline{\omega}$ and we consider the set of admissible domains

$$\mathcal{O}_k = \{\Omega = U \setminus \overline{\omega};\ \omega \text{ open set},\ \omega \subset V, \#\overline{\omega} \leq k\}. \tag{1}$$

For any $\Omega \in \mathcal{O}_k$, $k \geq 1$, the boundary of Ω can be splitted into $\partial\Omega = \Gamma_N \cup \partial U$ with $\Gamma_N = \partial\omega$. The boundary ∂U is divided in two components $\partial U = \Gamma_S \cup \Gamma_D$. The boundaries Γ_N and Γ_D shall receive Neumann and Dirichlet boundary conditions respectively, whereas Signorini conditions will be imposed on Γ_S. Let us point out that the open set ω is not necessarily a connected set as illustrated in Figure 1.

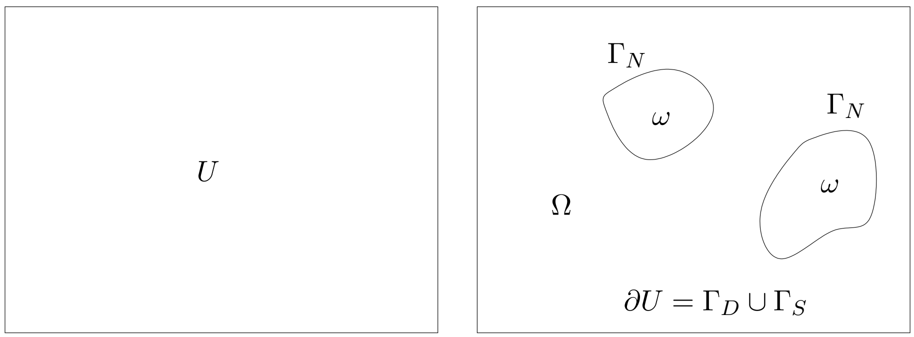

Figure 1. An admissible domain Ω

For $f \in C^\infty(\overline{U})$, we consider the following Signorini problem

$$\left\{\begin{array}{rcll} -\Delta u + u &=& f & \text{in} \quad \Omega, \\ u &=& 0 & \text{on} \quad \Gamma_D, \\ \partial_n u &=& 0 & \text{on} \quad \Gamma_N, \\ u \geq 0,\ \partial_n u \geq 0,\ u\partial_n u &=& 0 & \text{on} \quad \Gamma_S, \end{array}\right. \tag{2}$$

where n is the unit outwards normal vector to $\partial\Omega$ and ∂_n stands for the normal derivative on $\partial\Omega$. The Signorini problem (2) admits a unique weak solution $u(\Omega) \in K(\Omega)$ satisfying the variational inequality

$$\int_\Omega \nabla u \cdot \nabla(v - u)\, dx \geq \int_\Omega (f - u)(v - u)\, dx \qquad \forall v \in K(\Omega), \tag{3}$$

with

$$K(\Omega) = \{v \in H^1_{\Gamma_D}(\Omega)|\ v \geq 0 \text{ a.e. on } \Gamma_S\}, \tag{4}$$

and where $H^1_{\Gamma_D}(\Omega)$ stands for the classical Sobolev space of functions which belong to $H^1(\Omega)$ and with null trace on the boundary Γ_D.

Now, let us consider the energy functional

$$E(\Omega, u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right)\, dx - \int_\Omega fu\, dx. \tag{5}$$

Remark that the energy $E(\Omega, u)$ can also be written as

$$E(\Omega, u) = -\frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right)\, dx = -\frac{1}{2}\int_\Omega fu\, dx. \tag{6}$$

In this paper, we are interested in the functional

$$J(\Omega) = E(\Omega, u) + \lambda A(\Omega) - \mu P_c(\Omega)^2, \tag{7}$$

where $A(\Omega)$ and $P_c(\Omega)$ are defined by

$$A(\Omega) = |\Omega|, \tag{8}$$
$$P_c(\Omega) = \max(0, \mathcal{H}^1(\partial\Omega) - c). \tag{9}$$

In the above definitions, $|\Omega|$ denotes the Lebesgue measure of Ω in $\mathbb{R}^2$ and $\mathcal{H}^1(\partial\Omega)$ is the 1-dimensional measure of $\partial\Omega$. Constants λ and μ are positive and permit to take into account an area and a perimeter constraint respectively. The constant c is also positive and corresponds to a shift perimeter beyond which the perimeter constraint occurs. These constants have to be well chosen for the numerical problem.

For any $k \geq 1$, we are interested in the following shape optimization problem

$$\max\{J(\Omega) : \Omega \in \mathcal{O}_k\}. \tag{10}$$

Since the exterior boundary of Ω is fixed and equal to ∂U, the domains Ω satisfying (10) are actually determined by their internal boundary Γ_N. For the existence of an optimal domain satisfying (10) for the related linear problem obtained with $\Gamma_S = \emptyset$, we refer to [7].

In the following two sections, we study the shape sensitivity and the topological derivative of the functional J. Then we construct a levelset representation based on the shape derivative of J. This formulation provides a practical way to increase the shape functional J. The appearance of new hole is not possible with the only use of the levelset method based on the shape derivative. The use of topological derivative allows to create a new hole in order to increase J.

3 Shape derivative

The shape derivative for the Signorini problem can be calculated thanks to an abstract result of differentiability in convex sets (see [6]). A complete study of shape derivative can be found in [2], [3], or [25].

Let $\delta \geq 0$ be a given parameter and $\xi \in C_0^\infty(U)$ a given vector field. We consider the mapping $F_\delta = I + \delta\xi$ and we set $\Omega_\delta = F_\delta(\Omega)$. Since ξ has compact support in U, for small δ we have that $\Omega_\delta \subset U$ and $F_\delta(\partial U) = \partial U$ that is the exterior boundary of Ω is maintained fixed.

There exists a unique $u_\delta \in K_\delta(\Omega_\delta)$ solution of the following variational inequality: find $u_\delta \in K_\delta(\Omega_\delta)$ such that for all $v \in K_\delta(\Omega_\delta)$,

$$\int_{\Omega_\delta} \nabla u_\delta \cdot \nabla(v - u_\delta)\, dx \geq \int_{\Omega_\delta} (f - u_\delta)(v - u_\delta)\, dx \tag{11}$$

where

$$K_\delta(\Omega_\delta) = \{v \in H^1(\Omega_\delta) | \quad v = 0 \text{ a.e. on } \Gamma_D, \quad v \geq 0 \text{ a.e. on } \Gamma_S\}. \tag{12}$$

We assume that Ω is a smooth domain. It can be easily shown that the limit

$$dJ(\Omega, \xi) := \lim_{\delta \to 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} \tag{13}$$

exists and is equal to

$$dJ(\Omega; \xi) = \int_{\Gamma_N} (\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - fu)\langle \xi, n\rangle\, d\sigma + \int_{\Gamma_N} (\lambda - 2\mu P_c(\Omega)\mathcal{H})\langle \xi, n\rangle\, d\sigma, \tag{14}$$

where $\mathcal{H}$ is the curvature of the boundary Γ_N.

4 Topological derivative

For the sake of simplicity, we assume in this section that $\Omega = U$ (i.e. $\Gamma_N = \emptyset$) and that the boundary ∂U shall only receive the Signorini conditions (i.e. $\Gamma_D = \emptyset$), so that $\partial\Omega = \partial U = \Gamma_S$. Let us now consider the perforated domain $\Omega_\rho = U \setminus B_\rho$ where B_ρ is the ball of radius ρ, centered at x_0 and with boundary $\Gamma_\rho = \partial B_\rho$. In order to study the topological derivative of the functional J for the nonlinear Signorini problem, we need to compute the asymptotic expansion of the energy $E(\Omega_\rho, u_\rho)$ where u_ρ is the solution of the Signorini problem

$$\begin{cases} -\Delta u_\rho + u_\rho = f \quad \text{in } \Omega_\rho, \\ u_\rho \geq 0,\ \partial_n u_\rho \geq 0,\ u_\rho\, \partial_n u_\rho = 0 \quad \text{on } \Gamma_S, \\ \partial_n u_\rho = 0 \quad \text{on } \Gamma_\rho. \end{cases} \tag{15}$$

To this end, we shall make use of the so-called truncated domain technique (see [23],[8]). In this section we explain the main sketch of the technique used to compute the asymptotic expansion of the functional J. The detailed proof of the results of this section can be found in the appendix.

Now, we describe the truncated domain technique for the Signorini problem. Let us denote by Ω_R the domain

$$\Omega_R = U \setminus B_R, \tag{16}$$

where B_R is the ball of radius R with $R > \rho$, centered at x_0 and we define the ring $C(R, \rho)$ such that $\Omega_\rho = \Omega_R \cup \Gamma_R \cup C(R, \rho)$. We consider the following truncated problem

$$\begin{cases} -\Delta u_\rho^R + u_\rho^R = f \quad \text{in } \Omega_R, \\ u_\rho^R \geq 0,\ \partial_n u_\rho^R \geq 0,\ u_\rho^R\, \partial_n u_\rho^R = 0 \quad \text{on } \Gamma_S, \\ -\partial_n y_\rho + \partial_n u_\rho^R = A_\rho(u_\rho^R) \quad \text{on } \Gamma_R. \end{cases} \tag{17}$$

In the above problem, A_ρ is the Steklov-Poincaré operator defined by

$$\begin{array}{rccc} A_\rho: & H^{\frac{1}{2}}(\Gamma_R) & \to & H^{\frac{3}{2}}(\Gamma_R) \\ & v & \mapsto & \partial_n w_\rho \end{array} \tag{18}$$

where $w_\rho = w_\rho(v)$ is the unique solution of the problem

$$\begin{cases} -\Delta w_\rho + w_\rho = 0 \quad \text{in} \quad C(R, \rho), \\ w_\rho = v \quad \text{on} \quad \Gamma_R, \\ \partial_n w_\rho = 0 \quad \text{on} \quad \Gamma_\rho, \end{cases} \tag{19}$$

with $v \in H^{\frac{1}{2}}(\Gamma_R)$.

Finally, the function y_ρ appearing in problem (17) is the solution of the following problem

$$\begin{cases} -\Delta y_\rho + y_\rho = f_{|C(R,\rho)} \quad \text{in} \quad C(R, \rho), \\ y_\rho = 0 \quad \text{on} \quad \Gamma_R, \\ \partial_n y_\rho = 0 \quad \text{on} \quad \Gamma_\rho. \end{cases} \tag{20}$$

Then, the following result can be easily proved.

Proposition 1. *The solution u_ρ^R of problem (17) satisfies*

$$u_\rho^R = u_{\rho|\Omega_R}, \tag{21}$$

and we also have

$$u_{\rho|C(R,\rho)} = w_\rho(u_\rho^R) + y_\rho, \tag{22}$$

with w_ρ and y_ρ solutions of (19) and (20) respectively.

In order to obtain the topological derivative of J, we have to perform an expansion of the energy functional $E(\Omega_\rho, u_\rho)$ with respect to the radius ρ. Recall that (see (6))

$$E(\Omega_\rho, u_\rho) = -\frac{1}{2}\int_{\Omega_\rho} \left(|\nabla u_\rho|^2 + u_\rho^2\right) dx.$$

Using the domain truncation, we can split the integral in two parts, and we obtain

$$E(\Omega_\rho, u_\rho) = E(\Omega_R, u_\rho^R) - \frac{1}{2}E_\rho^{(1)}(u_\rho^R) + \frac{1}{2}E_\rho^{(2)}(f) \tag{23}$$

where

$$E(\Omega_R, u_\rho^R) = -\frac{1}{2}\int_{\Omega_R} \left(|\nabla u_\rho^R|^2 + (u_\rho^R)^2\right) dx \tag{24}$$

and

$$E_\rho^{(1)}(u_\rho^R) = \int_{C(R,\rho)} \left(|\nabla w_\rho|^2 + w_\rho^2\right) dx, \tag{25}$$

with $w_\rho = w_\rho(u_\rho^R)$ and

$$E_\rho^{(2)}(f) = -\int_{C(R,\rho)} \left(|\nabla y_\rho|^2 + y_\rho^2\right) dx. \tag{26}$$

Using an abstract result on conical differentiability of the solution of a variational inequation given in [6], we can adapt a result from [23] and show that the solution u_ρ^R of (2) on the truncated domain Ω_R admits the following expansion

$$u_\rho^R - u_0^R = O(\rho^2). \tag{27}$$

Actually, it can be proved that there exists a function q called the exterior topological derivative of solution u of the Signorini problem (2) such that

$$u_\rho^R = u_0^R + \rho^2 q + o(\rho^2).$$

This function q is the unique solution of a variational inequation (see [23] and [24] for details) and does not depend on ρ. Expansion (27) allows to perform the asymptotic expansion of (23) and we obtain (see the Appendix for the proof)

$$E(\Omega_\rho, u_\rho) = E(\Omega, u) - \left[\frac{u(x_0)^2}{2} + |\nabla u(x_0)|^2 - f(x_0)u(x_0)\right]\pi\rho^2 + o(\rho^2). \tag{28}$$

Now, using the expansions

$$\begin{aligned} A(\Omega_\rho) &= A(\Omega) - \pi\rho^2, & (29)\\ P_c(\Omega_\rho)^2 &= P_c(\Omega)^2 + 4\pi P_c(\Omega)\rho + o(\rho^2), & (30)\end{aligned}$$

we obtain the asymptotic expansion for J.

Theorem 1. *We have the following expansion of $J(\Omega_\rho)$:*

$$J(\Omega_\rho) = J(\Omega) - \left[\frac{u(x_0)^2}{2} + |\nabla u(x_0)|^2 - f(x_0)u(x_0) + \lambda\right]\pi\rho^2 - 4\mu\pi P_c(\Omega)\rho + o(\rho^2). \tag{31}$$

and the topological derivative $\mathcal{T}_\Omega(x_0)$ of the functional J at point $x_0 \in \Omega$ is given by

$$\begin{aligned} &\bullet\quad \mathcal{T}_\Omega(x_0) = -|\nabla u(x_0)|^2 - \tfrac{1}{2}u(x_0)^2 + uf(x_0) - \lambda, \quad \textit{if } P_c(\Omega) = 0,\\ &\bullet\quad \mathcal{T}_\Omega(x_0) = -4\mu P_c(\Omega), \quad \textit{if } P_c(\Omega) > 0. \end{aligned} \tag{32}$$

5 The levelset formulation

5.1 The Hamilton-Jacobi equation

The basic idea of the level set method is to represent a domain and its boundary as level sets of a continuous function ϕ defined on the whole domain U.

Let us consider the evolution of a domain $\Omega \subset U \subset \mathbb{R}^2$ under a velocity field ξ. More precisely, we define $\Omega_t = (I + t\xi)(\Omega)$, $t \in \mathbb{R}^+$, with a smooth vector field ξ. The domain and the boundary are defined by a function $\phi = \phi(x,t)$ such that

$$\Omega_t = \{x \in U,\ \phi(x,t) < 0\} \tag{33}$$

and

$$\partial\Omega_t = \{x \in U,\ \phi(x,t) = 0\}, \tag{34}$$

i.e. the boundary $\partial\Omega_t$ is the level curve of the function ϕ (see Fig. 2).

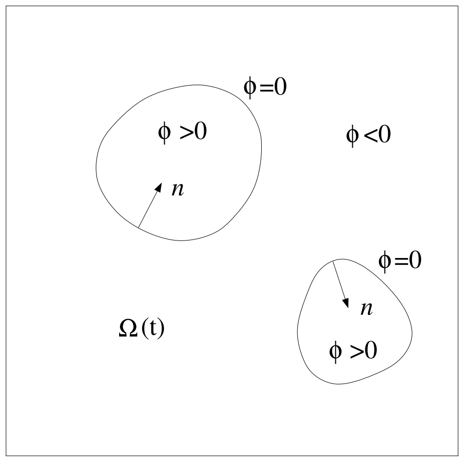

Figure 2. Domain and level set function.

Let $x(t)$ be the position of a particle on the boundary $\partial\Omega_t$ moving with velocity $\xi = \dot{x}(t)$. Differentiating the relation $\phi(x(t),t) = 0$ with respect to t, leads to the transport equation

$$\phi_t + \xi \cdot \nabla\phi = 0. \tag{35}$$

Moreover, the normal directions n to the level sets of ϕ are given by $n = \nabla\phi/|\nabla\phi|$. The evolution of ϕ is then governed by the Hamilton-Jacobi equation

$$\phi_t + \xi_n|\nabla\phi| = 0 \quad \text{in } U \times \mathbb{R}^+ \tag{36}$$

where ξ_n is the normal velocity (the normal component of V) i.e. $\xi_n = \langle \xi, n\rangle$. Initial data and boundary conditions have to be imposed together with the Hamilton-Jacobi equation (36). The initial data $\phi(0,x) = \phi_0(x)$ is choosen as the signed distance function to the initial boundary $\partial\Omega_0 = \partial\Omega$ i.e.

$$\phi_0(x) = \pm\ \mathrm{dist}(x, \partial\Omega_0), \tag{37}$$

with the minus (resp. plus) sign if the point x is inside (resp. outside) the initial domain $\Omega_0 = \Omega$.

Boundary condition has also to be imposed on the part of the boundary ∂U of the domain U where the normal velocity ξ_n is negative that is where the velocity is directed inwards the domain U. Alternatively, we decide to impose homogeneous Neumann boundary condition on the whole boundary ∂U :

$$\partial_n\phi = 0 \quad \text{on} \quad \partial U. \tag{38}$$

5.2 Normal velocity for the level set equation

When a hole is created inside the domain, the boundary condition for the state equation on the boundary of the hole is of Neumann type. The shape derivative is then given by (14). Since we locally (i.e. under small perturbations of the domain) require that $dJ(\Omega;\xi) > 0$, this leads to the following choice for the normal component $\xi_n = \langle \xi, n \rangle$ of the velocity field ξ:

$$\xi_n = \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - uf + \lambda - 2\mu P_c(\Omega)\mathcal{H} \quad \text{on} \quad \Gamma_N. \tag{39}$$

With a velocity field ξ satisfying (39), we clearly have $dJ(\Omega;\xi) > 0$ and then $J(\Omega_t) > J(\Omega)$ for t small enough.

6 The shape optimization algorithm

Let us now describe the steps of the general shape optimization algorithm.

First step: The initial domain
First of all, we choose an initial domain Ω^0 and we compute the solution of the Signorini problem (2) in Ω^0. This is done by the use of a finite element method on an appropriate (unstructured) mesh with an Uzawa algorithm for the treatment of the boundary constraint (see Section 7 below for details). Then, we compute the topological derivative $\mathcal{T}_{\Omega^0}(x)$ for all $x \in \Omega^0$, according to (32).

Second step: creating a hole
We use the topological derivative to create a hole in the domain Ω^0. More precisely, we find the point $x_0 \in \Omega^0$ such that $\mathcal{T}_{\Omega^0}(x_0) = \max_{x\in\Omega^0} \mathcal{T}_{\Omega^0}(x)$. If $\mathcal{T}_{\Omega^0}(x_0) > 0$ then we create a circular hole ω_ρ of radius $\rho > 0$, centered at x_0. We denote by Ω^0_* the new corresponding domain. Neumann condition will be imposed on the boundary of the new hole. Remark that the radius of this hole should be as small as possible, depending on the space step of the mesh.

Third step: evolution
Now we proceed to the evolution of the domain Ω^0_*. We need to compute the solution ϕ to the Hamilton-Jacobi equation (36)–(38). The initial ϕ is taken as the signed distance function to the domain Ω^0_*. According to (39), we compute the normal velocity ξ_n on the internal boundary part Γ_N of $\partial\Omega^0_*$. Remark that this requires the new computation of the solution of the Signorini problem in Ω^0_*. Since the normal velocity ξ_n is only known on the boundary part Γ_N, we need to extend it to the whole domain U. This is required in order to solve the level set equation (36) in U. The next section will explain how to proceed to construct the extended normal velocity in a numericaly accurate way.

Once we have computed the levelset function, we can determine the new domain Ω^1. Then go back to the first step with Ω^1 instead of Ω^0. Maybe a hole shouldn't be created at each step, but only when convergence for the shape derivative is obtained.

7 Numerical method for the Signorini problem

We use a piecewise linear finite element method with the Uzawa algorithm to compute the solution of the Signorini problem (2). The Uzawa algorithm is used for the non-negative boundary constraint on Γ_S. In the finite element framework, we are looking for $u_h \in \mathcal{U} \subset \mathbb{R}^n$ such that

$$E_h(u_h) = \inf_{v\in\mathcal{U}} E_h(v),$$

where $\mathcal{U} = \{v \in \mathbb{R}^n \mid Cv \geq 0\}$ and where the approximate energy functional

$$E_h(v) = \frac{1}{2}(Av, v) - (b, v)$$

corresponds to the finite element discretization of the energy functional

$$E(v) = \frac{1}{2}\left(\int_\Omega |\nabla v|^2 + v^2\right)\,dx - \int_\Omega fv\,dx.$$

In the above definition of E_h, the matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ is the usual stiffness-mass matrix associated to natural Neumann boundary conditions on Γ_S. The vector $b \in \mathbb{R}^n$ is related to the finite element discretization of the source term f. The matrix $C \in \mathcal{M}_{m\times n}(\mathbb{R})$ appearing in the definition of the space $\mathcal{U}$ has the effect to take the values of a vector on the only nodes of the boundary Γ_S (m is the number of nodes belonging to the Signorini boundary part Γ_S).

The Uzawa algorithm consists in the computation of a sequence $(u_h^k, \lambda_h^k) \in \mathbb{R}^n \times \mathbb{R}^+$, $k \geq 0$, defined by the following relations

- $Au_h^k - b - C^T\lambda_h^k = 0,$
- $\lambda_h^{k+1} = \max(\lambda_h^k - \rho(Cu_h^k)_i, 0) \quad$ for $1 \leq i \leq m$.

Under the condition that $0 < \rho < 2\lambda_1(A)/\|C\|^2$ where $\lambda_1(A)$ denotes the smallest eigenvalue of A, the sequence u_h^k converges to finite element approximation u_h of the Signorini problem (2).

8 Numerical method for the level set equation

Now we describe how to construct the extended normal velocity to the whole domain U and how to solve the related level set equation (36).

Let us start with a general remark for numericaly solving (36). For numerical accuracy, the solution of the level set equation (36) shouldn't be too flat or too steep. This is fulfilled for instance if ϕ is the distance function i.e. $|\nabla\phi| = 1$. Unfortunately, even if we start with a (signed) distance function for the initial data ϕ_0, the solution ϕ of the level set equation (36) does not generally remain close to a distance function. We can perform a reinitialization of ϕ at time t by solving the solution $\varphi = \varphi(\tau, x)$ of the following equation, up to the stationary state (see [18])

$$\begin{aligned} \varphi_\tau + S(\phi)(|\nabla\varphi| - 1) &= 0 \text{ in } \mathbb{R}^+ \times U, && (40)\\ \varphi(0,x) &= \phi(t,x),\ x \in U, && (41)\end{aligned}$$

Here, S is an approximation of the sign function i.e.

$$S(d) = \frac{d}{\sqrt{d^2 + |\nabla d|^2\varepsilon^2}} \tag{42}$$

with $\varepsilon = \min(\Delta x, \Delta y)$ where Δx and Δy stand for the space steps discretization in the x and y direction (see below). Another choices are possible for the approximate sign function. We refer to [18] for details.

8.1 Extended normal velocity

The normal velocity ξ_n must be defined on the whole domain U for solving the level set equation (36). Since the normal velocity ξ_n is only given on the boundary Γ_N (see (39)), we need to extend it to the domain U in order to solve the level set equation (36). Another reason for extanding the velocity is to enforce the solution ϕ of the level set equation to remain (close to) a distance function. Indeed, if we are able to compute an extended normal velocity ξ_{ext} such that

$$\nabla\xi_{\text{ext}} \cdot \nabla\phi = 0 \ \text{ in } \ U \times \mathbb{R}^+, \tag{43}$$

then it can be shown (see [27]) that the solution ϕ of the level set equation (36) satisfies $|\nabla\phi| = 1$. The way to construct the extension ξ_{ext} satisfying (43) at time t is to solve the following equation, up to the stationary state (see [16], [18])

$$\begin{aligned} q_\tau + S(\phi)\frac{\nabla\phi}{|\nabla\phi|}\cdot\nabla q &= 0 \ \text{ in } \ \mathbb{R}^+ \times D &\quad (44)\\ q(0,x) &= p(t,x),\ \ x \in D &\quad (45)\end{aligned}$$

where p equals to ξ_n on the boundary Γ_N and 0 elsewhere. The function S is the approximate sign function defined by (42).

8.2 Discretization of the level set equation

We fix U as the unit square $U = (0,1) \times (0,1)$. For the discretization of the Hamilton-Jacobi equation (36), we first define the mesh grid of U. We introduce the nodes P_{ij} whose coordinates are given by $(i\Delta x, j\Delta y)$ where Δx and Δy are the steps discretization in the x and y directions respectively. Let us also note $t^k = k\Delta t$ the discrete time for $k \in \mathbb{N}$, where Δt is the time step. We are seeking for an approximation $\phi_{ij}^k \simeq \phi(P_{ij}, t^k)$. The numerical scheme we use is proposed by Osher and Sethian [17],[19],[16]. This explicit upwind scheme reads as

$$\phi_{ij}^{k+1} = \phi_{ij}^k - \Delta t\ g(D_-^x\phi_{ij}^k, D_+^x\phi_{ij}^k, D_-^y\phi_{ij}^k, D_+^y\phi_{ij}^k) \tag{46}$$

where

$$D_-^x\phi_{ij} = \frac{\phi_{ij} - \phi_{i-1,j}}{\Delta x}, \quad D_+^x\phi_{ij} = \frac{\phi_{i+1,j} - \phi_{ij}}{\Delta x}, \tag{47}$$

are the backward and forward approximations of the x-derivative of ϕ at P_{ij}. Similar expressions hold for the approximations D_-^y and D_+^y of the y-derivative. The numerical flux g is given by

$$g_{ij} \ = \ g(D_-^x\phi_{ij}, D_+^x\phi_{ij}, D_-^y\phi_{ij}, D_+^y\phi_{ij}) = \max(v_{ij}, 0)\, G^+ + \min(v_{ij}, 0)\, G^-$$

with

$$\begin{aligned} G^+ = \ & \Big[\max(D_-^x v_{ij}, 0)^2 + \min(D_+^x v_{ij}, 0)^2 \\ & + \max(D_-^y v_{ij}, 0)^2 + \min(D_+^y v_{ij}, 0)^2\Big]^{1/2} \end{aligned}$$

$$\begin{aligned} G^- = \ & \Big[\min(D_-^x v_{ij}, 0)^2 + \max(D_+^x v_{ij}, 0)^2 \\ & + \min(D_-^y v_{ij}, 0)^2 + \max(D_+^y v_{ij}, 0)^2\Big]^{1/2} \end{aligned}$$

and $v_{ij} = \langle\xi_{\text{ext}}, n\rangle(P_{ij})$ is the extended normal velocity at point P_{ij}. This upwind scheme is stable under the CFL condition

$$(\max_U |\langle\xi_{\text{ext}}, n\rangle|)\Delta t\left(\frac{1}{\Delta x} + \frac{1}{\Delta y}\right) \le \frac{1}{2}. \tag{48}$$

8.3 Computing the extended velocity

At each iteration k of the previous scheme, we compute the extended normal velocity as the stationary solution of (44),(45). We compute $q_{ij}^n \simeq q(P_{ij}, t^n)$ from the following upwind approximation of (44) :

$$\begin{aligned} q_{ij}^{n+1} = q_{ij}^n - \Delta\tau\,[\ \ & \max(s_{ij}n_{ij}^x, 0)\, D_-^x q_{ij} + \min(s_{ij}n_{ij}^x, 0)\, D_+^x q_{ij} \\ & + \max(s_{ij}n_{ij}^y, 0)\, D_-^y q_{ij} + \min(s_{ij}n_{ij}^y, 0)\, D_+^y q_{ij}\ ], \end{aligned} \tag{49}$$

where $s_{ij} = S(\phi_{ij}^n)$. We use central differences to compute the approximation n_{ij} of the unit normal vector $n = (n^x, n^y) = (\phi_x/\sqrt{\phi_x^2 + \phi_y^2},\ \phi_y/\sqrt{\phi_x^2 + \phi_y^2})$ at node P_{ij}. The initial value q_0 is equal to V_n on the grid points with a distance to the interface less than $\min(\Delta x, \Delta y)$ and equals zero elsewhere.

9 Numerical results

We present numerical computations performed with $\lambda = 0.3,\ \mu = 0.001,\ c = 0.6$. The source term f is chosen with compact support in U and is given by (see Figure 3):

$$f = \begin{cases} 10 & \text{in } [0.2, 0.4]^2, \\ -10 & \text{in } [0.6, 0.8]^2, \\ 0 & \text{elsewhere.} \end{cases}$$

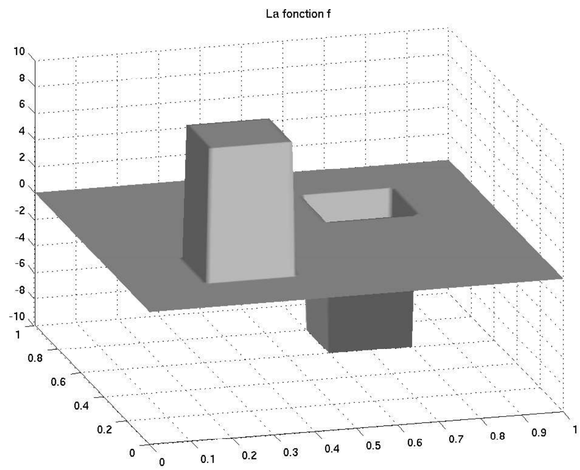

Figure 3. The source function f

The numerical solution u and its gradient are represented on Figure 4. We observe that the

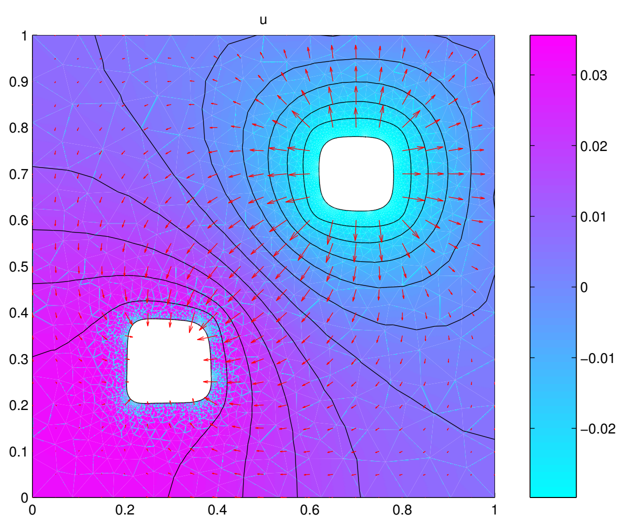

Figure 4. The solution u and its gradient on the optimal domain

functional J is converging to a maximum which is equal to 0.2798122, while the domain is also converging to an optimal domain Ω (see Figure 5).

Nevertheless, the evolution is very sensitive to the small perturbations as we can see from the oscillations of the energy functional during the iterations. This is due to the fact that the shape derivative is only a directional derivative for the nonlinear Signorini problem while it is a Fréchet derivative for the related linear problem obtained with $\Gamma_S = \emptyset$.

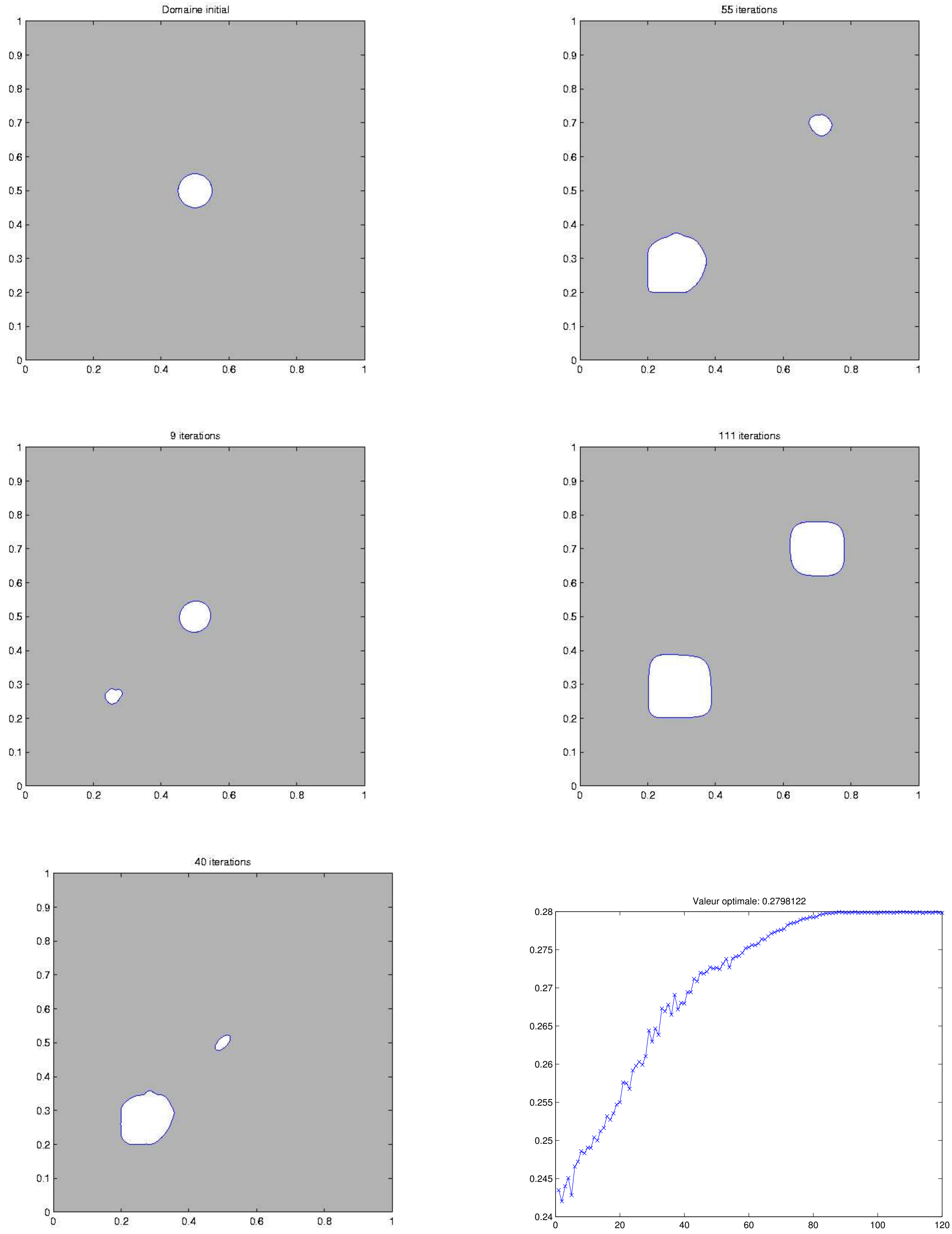

Figure 5. Evolution of the domain and shape functional J

10 Appendix: The topological derivative with Neumann conditions on the hole

We present the results which are used in order to determine the topological derivatives of the energy functional. The technique is proposed in [23], [8]. The proof of the asymptotic expansion of Steklov-Poincaré operator is given in [24]. For the convenience of the reader we present complete arguments.

The domain decomposition technique can be described in the following way. The actual domain is divided into two parts $\Omega_\rho = \Omega_R \cup \Gamma_R \cup C(R,\rho)$. In the ring $C(R,\rho)$ the singular perturbation of the domain is located, the moving part of its boundary Γ_ρ for $\rho > 0$ is the small parameter. On the other part of its boundary Γ_R the Steklov-Poincaré operator is defined and the asymptotics of the operator are determined in function of the parameter ρ. The second domain Ω_R of the decomposition depends only on the parameter $\rho > 0$ by the nonlocal boundary conditions prescribed in terms of the Steklov-Poincaré operator A_ρ, so we have the regular perturbation of the boundary conditions for the nonlinear boundary value problem in Ω_R. The conical differentiability of solutions to the variational inequality in Ω_R is shown and results in the asymptotic expansion of the energy functional for our shape optimization problem under investigations.

Now we give in the forthcoming subsections, the asymptotic expansions of the energy terms $E_\rho^{(1)}(v)$, $E_\rho^{(2)}(f)$ and $E(\Omega_\rho, u_\rho)$ appearing in the relation (23). For the sake of simplicity, we will assume in what follows that $x_0 = 0$.

10.1 Asymptotic expansion of $E_\rho^{(1)}(v)$

To begin with, for any v in $H^{\frac{1}{2}}(\Gamma_R)$, we consider the following problem (see (19))

$$\left\{ \begin{array}{rcll} -\Delta w_\rho + w_\rho & = & 0 & \text{in} \quad C(R,\rho), \\ w_\rho & = & v & \text{on} \quad \Gamma_R, \\ \partial_n w_\rho & = & 0 & \text{on} \quad \Gamma_\rho. \end{array} \right. \tag{50}$$

The Steklov-Poincaré operator A_ρ is defined in the following way

$$\begin{array}{rccc} A_\rho : & H^{\frac{1}{2}}(\Gamma_R) & \to & H^{\frac{3}{2}}(\Gamma_R) \\ & v & \mapsto & \partial_n w_\rho \end{array} \tag{51}$$

Since $v \in H^{\frac{1}{2}}(\Gamma_R)$ we can write v in the form of Fourier series, with (r,ϕ) the polar coordinates at the origin

$$v(\phi) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \sin(k\phi) + b_k \cos(k\phi))$$

whose coefficients verify:

$$\sum_{k=1}^{\infty} \sqrt{1+k^2}(a_k^2 + b_k^2) \leq M,$$

where M is a constant depending only on R. This implies two important properties:

$$\sum_{k=1}^{\infty} (a_k^2 + b_k^2) \leq M, \qquad \sum_{k=1}^{\infty} k(a_k^2 + b_k^2) \leq M. \tag{52}$$

The following result gives the asymptotic expansion of the energy term

$$E_\rho^{(1)}(v) = \int_{C(R,\rho)} \left(|\nabla w_\rho|^2 + w_\rho^2 \right) dx \tag{53}$$

where $w_\rho = w_\rho(v)$ is the solution of (50).

Theorem 2. *The energy $E_\rho^{(1)}(v)$ admits the expansion*

$$E_\rho^{(1)}(v) = E^{(1)}(v) - \left(\frac{\pi(a_1^2+b_1^2)}{2I_1(R)^2} + \frac{\pi a_0^2}{4I_0(R)^2}\right)\rho^2 + \mathfrak{R}(v),$$

with $E^{(1)}(v) = E_0^{(1)}(v)$ and $\mathfrak{R}(v) = o(\rho^2)$ uniformly on bounded subsets of $H^1(\Omega_R)$. The Bessel functions I_0 and I_1 are defined in (60).

Proof. Since every compact can be covered by a finite number of balls, it is enough to prove the lemma for a fixed ball in $H^1(\Omega_R)$. Thus we can assume that (52) occurs. The proof consists in obtaining explicit formulae for w and w_ρ in series. Then we can calculate energies explicitely and obtain an upper bound for the rest $\mathfrak{R}(v)$.

We look for the solution w_ρ of (50) in $C(R,\rho)$ with the form

$$w_\rho(r,\phi) = \frac{1}{2}a_0c_0(r) + \sum_{k=1}^{\infty} c_{k,\rho}(r)(a_k\sin(k\phi) + b_k\cos(k\phi)). \tag{54}$$

The Laplacian in polar coordinates is equal to:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\phi^2}$$

Inserting equation (54) in $-\Delta w_\rho + w_\rho = 0$ we obtain for $k \geq 1$,

$$c''_{k,\rho}(r) + \frac{1}{r}c'_{k,\rho}(r) - (\frac{k^2}{r^2}+1)c_{k,\rho}(r) = 0, \tag{55}$$

and

$$c''_{0,\rho}(r) + \frac{1}{r}c'_{0,\rho}(r) - c_{0,\rho}(r) = 0. \tag{56}$$

Multiplying (55) and (56) by r^2, we get, for $k \geq 1$

$$r^2c''_{k,\rho}(r) + rc'_{k,\rho}(r) - (k^2+r^2)c_{k,\rho}(r) = 0, \tag{57}$$

and

$$r^2c''_{0,\rho}(r) + rc'_{0,\rho}(r) - r^2c_{0,\rho}(r) = 0. \tag{58}$$

According to [26], the solutions of equations (57) and (58) are given by

$$c_k(r) = A_kI_k(r) + B_kK_k(r) \qquad k \geq 0, \tag{59}$$

where A_k,$B_k \in \mathbb{R}$ and where I_k, K_k are Bessel functions defined by

$$I_k(r) = \sum_{m=0}^{\infty}\frac{(\frac{r}{2})^{k+2m}}{m!(k+m)!}, \qquad k \geq 0, \tag{60}$$

and for $k \geq 1$

$$K_k(r) = \frac{1}{2}\sum_{m=0}^{k-1}\frac{(-1)^m(k-m-1)!}{m!(\frac{r}{2})^{k-2m}} + \sum_{m=0}^{\infty}\frac{(-1)^{k+1}(\frac{r}{2})^{k+2m}}{m!(k+m)!}\tilde{K}_m(r) \tag{61}$$

with

$$\tilde{K}_m(r) = \left[\ln\left(\frac{r}{2}\right) - \frac{1}{2}\psi(m+1) - \frac{1}{2}\psi(k+m+1)\right],$$

and ψ is the logarithmic derivative of fonction Γ, i.e.

$$\psi(x) = \frac{\partial}{\partial h} \ln \Gamma(x+h). \tag{62}$$

Finally for $k = 0$

$$K_0(r) = -\ln\left(\frac{r}{2}\right) I_0(r) + \sum_{m=0}^{\infty} \frac{(\frac{r}{2})^{2m}}{m!^2} \psi(m+1). \tag{63}$$

The boundary conditions on Γ_R and Γ_ρ allow us to obtain the following systems for $k \geq 0$

$$\begin{aligned} A_k I_k(R) + B_k K_k(R) &= 1 \\ A_k I_k'(\rho) + B_k K_k'(\rho) &= 0. \end{aligned}$$

We deduce the expression of $c_{k,\rho}(r)$ for $k \geq 0$ in a form suited to the asymptotic expansion when $\rho \to 0$:

$$c_{k,\rho}(r) = \frac{I_k(r)}{I_k(R)} + \dot{c}_{k,\rho}(r) \tag{64}$$

with

$$\dot{c}_{k,\rho}(r) = -\frac{\frac{I_k'(\rho)}{K_k'(\rho)}}{\frac{I_k(R)}{K_k(R)} - \frac{I_k'(\rho)}{K_k'(\rho)}} \left[\frac{K_k(r)}{K_k(R)} - \frac{I_k(r)}{I_k(R)}\right].$$

For $k \geq 1$, we deduce from formulas (60) and (61) that

$$I_k'(\rho) = \frac{\rho^{k-1}}{2^k(k-1)!} + O(\rho^{k+1}), \tag{65}$$

$$K_k'(\rho) = -\frac{k! 2^{k-1}}{\rho^{k+1}} + o(\rho^{-k-1}), \tag{66}$$

so that

$$\frac{I_k'(\rho)}{K_k'(\rho)} = -\frac{\rho^{2k}}{k!(k-1)! 2^{2k-1}} + o(\rho^{2k}), \quad k \geq 1. \tag{67}$$

In particular, we get

$$\frac{I_1'(\rho)}{K_1'(\rho)} = -\frac{\rho^2}{2} + o(\rho^2). \tag{68}$$

For $k = 0$, we deduce from formulas (60) and (63) that

$$I_0'(\rho) = \frac{\rho}{2} + O(\rho^3), \quad K_0'(\rho) = -\frac{1}{\rho} + o(\rho). \tag{69}$$

Thus we have

$$\frac{I_0'(\rho)}{K_0'(\rho)} = -\frac{\rho^2}{2} + o(\rho^2). \tag{70}$$

The function w_ρ can be extended as:

$$w_\rho = w + z_\rho, \tag{71}$$

where w is the solution of problem (50) for $\rho = 0$ given by

$$w(r, \phi) = \frac{1}{2} a_0 \frac{I_0(r)}{I_0(R)} + \sum_{k=1}^{\infty} \frac{I_k(r)}{I_k(R)} (a_k \sin(k\phi) + b_k \cos(k\phi)), \tag{72}$$

and

$$z_\rho(r, \phi) = \frac{a_0}{2} \dot{c}_{0,\rho}(r) + \sum_{k=1}^{\infty} \dot{c}_{k,\rho}(r)(a_k \sin(k\phi) + b_k \cos(k\phi)). \tag{73}$$

Using (71) in (53), we get

$$E_\rho^{(1)}(v) = \int_{C(R,\rho)} \left(|\nabla w + \nabla z_\rho|^2 + (w + z_\rho)^2\right) dx \tag{74}$$

and then

$$E_\rho^{(1)}(v) = E^{(1)}(v) + I_1 + I_2 + I_3 \tag{75}$$

with

$$I_1 = \int_{C(R,\rho)} \left((\partial_r z_\rho)^2 + \frac{1}{r^2}(\partial_\phi z_\rho)^2 + z_\rho^2\right) dx, \tag{76}$$

$$I_2 = 2\int_{C(R,\rho)} \left(\partial_r w\, \partial_r z_\rho + \frac{1}{r^2}\partial_\phi w\, \partial_\phi z_\rho + w z_\rho\right) dx, \tag{77}$$

$$I_3 = -\int_{B(\rho)} \left(|\nabla w|^2 + w^2\right) dx. \tag{78}$$

We first deal with the integral term I_2. We have

$$I_2 = 2\int_{C(R,\rho)} \left(\langle \nabla w, \nabla z_\rho\rangle + w z_\rho\right) dx. \tag{79}$$

The function w satisfies $-\Delta w + w = 0$ in $C(R,\rho)$. Moreover, according to (71) and (50), we have that $z_\rho = 0$ on Γ_R and $\partial_n w = -\partial_n z_\rho$ on Γ_ρ. Then we obtain that

$$I_2 = -2\int_{\Gamma_\rho} z_\rho \partial_n z_\rho \, d\sigma. \tag{80}$$

Since n is the outer normal vector to $C(R,\rho)$, we have $\partial_n z_\rho = -\partial_r z_\rho$ on Γ_ρ. Thus, from expression (73) of z_ρ, and in view of expansions (65),(66) and (69), we can show that the main term in I_2 is given by

$$2\pi\rho(a_1^2 + b_1^2)\frac{d\dot{c}_{1,\rho}}{dr}(\rho)\dot{c}_{1,\rho}(\rho) = -\frac{2\pi(a_1^2 + b_1^2)}{4I_1(R)^2}\rho^2 + o(\rho^2).$$

Thus we obtain

$$I_2 = -\frac{2\pi(a_1^2 + b_1^2)}{4I_1(R)^2}\rho^2 + o(\rho^2). \tag{81}$$

Now we turn to the integral I_3. We have

$$I_3 = -\int_{B(\rho)} \left(|\nabla w|^2 + w^2\right) dx = -\int_{\Gamma_\rho} w \partial_r w \, d\sigma. \tag{82}$$

Then it can be shown that

$$I_3 = -\left(\frac{\pi a_0^2}{4I_0(R)^2} + \frac{\pi(a_1^2 + b_1^2)}{4I_1(R)^2}\right)\rho^2 + o(\rho^2). \tag{83}$$

Before calculating I_1, we will make some remarks. First of all from (61), for $k \geq 1$

$$\int_\rho^R r K_k'(r)^2 dr = O(\rho^{-2k}), \tag{84}$$

and for $k = 0$, we get

$$\int_\rho^R r K_0'(r)^2 dr = O(\ln \rho), \tag{85}$$

Then from (60), for $k \geq 0$,

$$\int_{\rho}^{R} r I_k'(r)^2 dr = O(1). \tag{86}$$

We divide now I_1 in two parts $I_1 = I_{11} + I_{12}$ with

$$\begin{aligned} I_{11} &= \int_{C(R,\rho)} \left((\partial_r z_\rho)^2 + \frac{1}{r^2}(\partial_\phi z_\rho)^2 \right) dx, \\ I_{12} &= \int_{C(R,\rho)} z_\rho^2 \, dx. \end{aligned}$$

As a consequence, since $\frac{I_k'(\rho)}{K_k'(\rho)} = O(\rho^{2k})$ for $k \geq 1$, and since $\frac{I_0'(\rho)}{K_0'(\rho)} = O(\rho^2)$ the main terms in I_{11} coming from $(\partial_r z_\rho)^2$ and $\frac{1}{r^2}(\partial_\phi z_\rho)^2$ are given respectively by

$$\int_0^{2\pi} \int_\rho^R \left(\frac{d\dot{c}_{1,\rho}}{dr}(r) \right)^2 (a_1^2 \sin^2(k\phi) + b_1^2 \cos^2(k\phi))\, r dr d\theta$$

and

$$\int_0^{2\pi} \int_\rho^R \frac{1}{r^2} \dot{c}_{k,\rho}(r)^2 (a_1^2 \cos^2(k\phi) + b_1^2 \sin^2(k\phi))\, r dr d\theta.$$

The computation of these two terms leads to

$$I_{11} = \frac{\pi(a_1^2 + b_1^2)}{4 I_1(R)^2} \rho^2 + o(\rho^2). \tag{87}$$

We also show easily that

$$I_{12} = o(\rho^2). \tag{88}$$

Now we can conclude with (87), (88), (81) and (83) that

$$E_\rho^{(1)}(u) = E^{(1)}(v) - \left(\frac{\pi(a_1^2 + b_1^2)}{2 I_1(R)^2} + \frac{\pi a_0^2}{4 I_0(R)^2} \right) \rho^2 + o(\rho^2). \tag{89}$$

The proof of Theorem (2) is then complete. ■

10.2 Asymptotic expansion of $E_\rho^{(2)}(f)$

We consider the problem (see (20))

$$\begin{cases} -\Delta y_\rho + y_\rho &= f_{|C(R,\rho)} \quad \text{in} \quad C(R,\rho) \\ y_\rho &= 0 \quad \text{on} \quad \Gamma_R \\ \partial_n y_\rho &= 0 \quad \text{on} \quad \Gamma_\rho \end{cases} \tag{90}$$

with $f \in C^\infty(\mathbb{R}^2)$. We will study the following function:

$$f_{|C(R,\rho)} \mapsto \frac{\partial y_\rho}{\partial n}_{|\Gamma_R} = g_\rho.$$

We would like to obtain an expansion of g_ρ with respect to ρ. We have the following Fourier expansion for f.

$$f(r,\phi) = \frac{1}{2}\tilde{a}_0(r) + \sum_{k=1}^{\infty} (\tilde{a}_k(r) \sin(k\phi) + \tilde{b}_k(r) \cos(k\phi)).$$

We have the following theorem:

Theorem 3. *The function g_ρ admits the expansion*

$$g_\rho = g_0 - \left(\frac{h_0^a(R) - \tilde{a}_0(0)}{4RI_0(R)}\right)\rho^2 - \left(\frac{h_1^a(R)}{2RI_1(R)}\sin\phi + \frac{h_1^b(R)}{2RI_1(R)}\cos\phi\right)\rho^2 + o(\rho^2). \tag{91}$$

where $h_0^a(R)$, $h_1^a(R)$ are defined in (105) *and $h_1^b(R)$ is obtained from $h_1^a(R)$ substituing $\tilde{b}_1(t)$ to $\tilde{a}_1(t)$ in* (105).

Proof. We look for the solution y_ρ of problem (90) in the form

$$y_\rho = \frac{1}{2}c_{0,\rho}(r) + \sum_{k=1}^{\infty} c_{k,\rho}(r)\sin(k\phi) + d_{k,\rho}(r)\cos(k\phi). \tag{92}$$

Inserting (92) into $-\Delta y_\rho + y_\rho = f_{|C(R,\rho)}$, we obtain, for $k \geq 0$,

$$r^2 c''_{k,\rho}(r) + rc'_{k,\rho}(r) - (k^2 + r^2)c_{k,\rho}(r) = -r^2\tilde{a}_k(r), \tag{93}$$

and, for ≥ 1,

$$r^2 d''_{k,\rho}(r) + rd'_{k,\rho}(r) - (k^2 + r^2)d_{k,\rho}(r) = -r^2\tilde{b}_k(r). \tag{94}$$

Let us deal first with the coefficients $c_{k,\rho}$. We solve (93) to get (see [7] for details), for $k \geq 0$,

$$c_{k,\rho}(r) = A_k(r,\rho)I_k(r) + B_k(r,\rho)K_k(r) \tag{95}$$

with

$$A_k(r,\rho) = L_A(r,k) + \alpha_k(\rho), \tag{96}$$

$$B_k(r,\rho) = L_B(r,k) + \beta_k(\rho), \tag{97}$$

and

$$L_A(r,k) = -\int_R^r t\tilde{a}_k(t)K_k(t)\,dt, \tag{98}$$

$$L_B(r,k) = \int_R^r t\tilde{a}_k(t)I_k(t)\,dt. \tag{99}$$

The boundary conditions for $c_{k,\rho}$ are now

$$c_{k,\rho}(R) = 0, \quad c'_{k,\rho}(\rho) = 0.$$

This leads to the following expression for α_k and β_k:

$$\alpha_k(\rho) = \frac{L_A(\rho,k)\frac{I'_k(\rho)}{K'_k(\rho)} + L_B(\rho,k)}{\frac{I_k(R)}{K_k(R)} - \frac{I'_k(\rho)}{K'_k(\rho)}}, \tag{100}$$

$$\beta_k(\rho) = -\alpha_k(\rho)\frac{I_k(R)}{K_k(R)}. \tag{101}$$

For $k \geq 2$, the expansions of $\alpha_k(\rho)$ and $\beta_k(\rho)$ provide terms of order strictly greater than ρ^2. So, we only have to deal with the cas $k = 1$ and $k = 0$. From (100) and expansions (68) and (70), we get for $i = 0, 1$

$$\alpha_i(\rho) = \frac{K_i(R)}{I_i(R)}\left(L_A(\rho,i)\frac{I'_i(\rho)}{K'_i(\rho)} + L_B(\rho,i)\right) \times \left(1 + \frac{K_i(R)}{I_i(R)}\frac{I'_i(\rho)}{K'_i(\rho)} + O\left(\frac{I'_i(\rho)^2}{K'_i(\rho)^2}\right)\right), \tag{102}$$

and

$$\alpha_i(\rho) = \alpha_i(0) - \frac{K_i(R)h_i^a(R)}{2I_i(R)}\rho^2 + \frac{K_i(R)}{I_i(R)}\int_0^\rho t\tilde{a}_i(t)I_i(t)\,dt + o(\rho^2), \tag{103}$$

with
$$\alpha_i(0) = -\frac{K_i(R)}{I_i(R)} \int_0^R t\tilde{a}_i(t) I_i(t)\, dt, \tag{104}$$
and
$$h_i^a(R) = -\frac{K_i(R)}{I_i(R)} \int_0^R t\tilde{a}_i(t) I_i(t)\, dt + \int_0^R t\tilde{a}_i(t) K_i(t)\, dt. \tag{105}$$
We also have thanks to (101) the expansion
$$\beta_i(\rho) = \beta_i(0) + \frac{h_i^a(R)}{2}\rho^2 - \int_0^\rho t\tilde{a}_i(t) I_i(t)\, dt + o(\rho^2),$$
with
$$\beta_i(0) = \int_0^R t\tilde{a}_i(t) I_i(t)\, dt. \tag{106}$$
In (103), the term $\int_0^\rho t\tilde{a}_i(t) I_i(t)$ gives, for $i = 0$,
$$\int_0^\rho t\tilde{a}_0(t) I_0(t) = \frac{\tilde{a}_0(0)}{2}\rho^2 + O(\rho^3). \tag{107}$$
On the contrary, for $i = 1$, we get due to (60)
$$\int_0^\rho t\tilde{a}_1(t) I_1(t) = O(\rho^3). \tag{108}$$
Thus we have
$$\alpha_0(\rho) = \alpha_0(0) - \frac{K_0(R)(h_0^a(R) - \tilde{a}_0(0))}{2I_0(R)}\rho^2 + o(\rho^2), \tag{109}$$
and
$$\alpha_1(\rho) = \alpha_1(0) - \frac{K_1(R) h_1^a(R)}{2I_1(R)}\rho^2 + o(\rho^2). \tag{110}$$
Now we can write the expansion of $c'_{k,\rho}(R)$ with respect to the small parameter ρ. We have
$$c'_{k,\rho}(R) = \alpha_k(\rho) I'_k(R) + \beta_k(\rho) K'_k(R).$$
Thus we obtain the expansions
$$c'_{0,\rho}(R) = c'_{0,0}(R) - \frac{h_0^a(R) - \tilde{a}_0(0)}{2}\left(\frac{K_0(R)}{I_0(R)} I'_0(R) - K'_0(R)\right)\rho^2 + o(\rho^2), \tag{111}$$
$$c'_{1,\rho}(R) = c'_{1,0}(R) - \frac{h_1^a(R)}{2}\left(\frac{K_1(R)}{I_1(R)} I'_1(R) - K'_1(R)\right)\rho^2 + o(\rho^2), \tag{112}$$
and for $k \geq 2$,
$$c'_{k,\rho}(R) = c'_{k,0}(R) + o(\rho^2). \tag{113}$$
Finally, we obtain
$$c'_{0,\rho}(R) = c'_{0,0}(R) - \left(\frac{h_0^a(R) - \tilde{a}_0(0)}{2RI_0(R)}\right)\rho^2 + o(\rho^2). \tag{114}$$
$$c'_{1,\rho}(R) = c'_{1,0}(R) - \frac{h_1^a(R)}{2RI_1(R)}\rho^2 + o(\rho^2). \tag{115}$$
Concerning the expansion of $d'_{k,\rho}(R)$, we obtain exactly the same results by putting the coefficients $\tilde{a}_k(r)$ in place of the coefficients $\tilde{b}_k(r)$ in expressions (114) and (115).

Deriving the Fourier series we obtain the following expansion
$$\frac{\partial y_\rho}{\partial n}_{|\Gamma_R} = \frac{1}{2} c'_{0,\rho}(R) + \sum_{k=1}^{\infty} c'_{k,\rho}(R) \sin(k\phi) + d'_{k,\rho}(R)\cos(k\phi). \tag{116}$$

Inserting in (116) the obtained expansions for $c'_{0,\rho}(R)$, $c'_{k,\rho}(R)$ and $d'_{k,\rho}(R)$ we get

$$g_\rho = g_0 - \left(\frac{h_0^a(R) - \tilde{a}_0(0)}{4RI_0(R)}\right)\rho^2 - \left(\frac{h_1^a(R)}{2RI_1(R)}\sin\phi + \frac{h_1^b(R)}{2RI_1(R)}\cos\phi\right)\rho^2 + o(\rho^2). \tag{117}$$

The expansion (91) is then proved. ■

We are now in position to compute the asymptotic expansion for the energy term $E_\rho^{(2)}(f) = -\int_{C(R,\rho)}\left(|\nabla y_\rho|^2 + y_\rho^2\right)dx$.

Theorem 4. *The energy $E_\rho^{(2)}(f)$ has the expansion*

$$E_\rho^{(2)}(f) = E^{(2)}(f) - \frac{\pi h_0^a(R)^2}{4}\rho^2 - \frac{\pi(h_1^a(R)^2 + h_1^b(R)^2)}{2}\rho^2 + o(\rho^2). \tag{118}$$

Proof. Thanks to the previous asymptotic expansions, we obtain

$$c_{0,\rho}(r) = c_{0,0}(r) - \frac{h_0^a(R) - \tilde{a}_0(0)}{2I_0(R)}M_0(r)\rho^2 + o(\rho^2), \tag{119}$$

$$c_{1,\rho}(r) = c_{1,0}(r) - \frac{h_1^a(R)}{2I_1(R)}M_1(r)\rho^2 + o(\rho^2), \tag{120}$$

where for $i = 0, 1$, $h_i^a(R)$ is given by (105) and

$$c_{0,0}(r) = K_0(r)\int_0^r t\tilde{a}_0(t)I_0(t)\,dt - I_0(r)\int_R^r t\tilde{a}_0(t)K_0(t)\,dt - \frac{K_0(R)I_0(r)}{I_0(R)}\int_0^R t\tilde{a}_0(t)I_0(t)\,dt, \tag{121}$$

$$M_i(r) = K_i(R)I_i(r) - K_i(r)I_i(R).$$

For $k \geq 2$, we clearly have

$$c_{k,\rho}(r) = c_{k,0}(r) + o(\rho^2), \tag{122}$$

$$d_{k,\rho}(r) = d_{k,0}(r) + o(\rho^2), \tag{123}$$

We are able now to give the expansion of the energy. Using the Green formula, we obtain

$$E_\rho^{(2)}(f) = -\int_{C(R,\rho)} f y_\rho\,dx. \tag{124}$$

Then we replace y_ρ with its expansion in Fourier series (92) and we get

$$E_\rho^{(2)}(f) = -\frac{1}{2}\int_{C(R,\rho)} fc_{0,\rho}\,dx - \sum_{k=1}^{\infty}\int_{C(R,\rho)}\left(fc_{k,\rho}\sin(k\phi) + fd_{k,\rho}\cos(k\phi)\right)dx. \tag{125}$$

From (119) and substituing f with its expansion in Fourier series, we have that

$$\begin{aligned}
\int_{C(R,\rho)} fc_{0,\rho}\,dx &= \pi\int_\rho^R r\tilde{a}_0(r)c_{0,\rho}(r)\,dr \\
&= \pi\int_\rho^R r\tilde{a}_0(r)c_{0,0}(r)\,dr + \frac{\pi(h_0^a(R)^2 - \tilde{a}_0(0)h_0^a(R))}{2}\rho^2 + o(\rho^2) \\
&= \pi\int_0^R r\tilde{a}_0(r)c_{0,0}(r)\,dr + \frac{\pi(h_0^a(R)^2 - 2\tilde{a}_0(0)h_0^a(R))}{2}\rho^2 + o(\rho^2).
\end{aligned}$$

We also have

$$
\begin{aligned}
\int_{C(R,\rho)} f c_{1,\rho} \sin\phi &= \pi \int_{\rho}^{R} r\tilde{a_1}(r) c_{1,\rho}(r)\, dr \\
&= \pi \int_{\rho}^{R} r\tilde{a_1}(r) c_{1,0}(r)\, dr + \frac{\pi h_1^a(R)^2}{2}\rho^2 + o(\rho^2) \\
&= \pi \int_{0}^{R} r\tilde{a_1}(r) c_{1,0}(r)\, dr + \frac{\pi (h_1^a(R)^2 + \tilde{a_1}(0) h_1^a(R))}{2}\rho^2 + o(\rho^2).
\end{aligned}
$$

From the expansion in Fourier series of f, we clearly have $\tilde{a_1}(0) = 0$. Now we can conclude, thanks to (122),(123) that

$$
E_\rho^{(2)}(f) = E^{(2)}(f) - \frac{\pi(h_0^a(R)^2 - 2\tilde{a}_0(0) h_0^a(R))}{4}\rho^2 - \frac{\pi(h_1^a(R)^2 + h_1^b(R)^2)}{2}\rho^2 + o(\rho^2). \tag{126}
$$

The proof of the theorem 4 is then complete. ■

10.3 Asymptotic expansion of $E(\Omega_\rho, u_\rho)$

The energy term $E_\rho = E(\Omega_\rho, u_\rho)$ is given by (see (23)):

$$
E_\rho = -\frac{1}{2}\int_{\Omega_R} \left(|\nabla u_\rho^R|^2 + (u_\rho^R)^2\right) dx - \frac{1}{2}E_\rho^{(1)}(u_\rho^R) + \frac{1}{2}E_\rho^{(2)}(f). \tag{127}
$$

Using Theorem 2 and Theorem 4, we obtain

$$
\begin{aligned}
E_\rho - E_0 &= \int_{\Gamma_R} A_\rho(u_\rho^R) u_0^R \, d\sigma - \int_{\Gamma_R} A_0(u_\rho^R) u_0^R \, d\sigma \\
&\quad - \frac{\pi(h_0^a(R)^2 - 2\tilde{a}_0(0) h_0^a(R))}{8}\rho^2 - \frac{\pi(h_1^a(R)^2 + h_1^b(R)^2)}{4}\rho^2 \\
&\quad + \left(\frac{\pi(a_1^2 + b_1^2)}{4I_1(R)^2} + \frac{\pi a_0^2}{8I_0(R)^2}\right)\rho^2 + \int_{\Gamma_R} u_0^R \partial_n (y_\rho - y_0) d\sigma + o(\rho^2).
\end{aligned}
$$

From formulas (72),(73) we have

$$
\begin{aligned}
\int_{\Gamma_R} A_\rho(u_\rho^R) u_0^R \, d\sigma - \int_{\Gamma_R} A_0(u_\rho^R) u_0^R \, d\sigma &= \int_{\Gamma_R} (A_\rho - A_0)(u_0^R) u_0^R \, d\sigma + o\left(\rho^2\right) && (128) \\
&= \int_{\Gamma_R} w(u_0^R) \partial_n z_\rho(u_0^R)\, d\sigma + o\left(\rho^2\right) && (129) \\
&= -\frac{\pi a_0^2}{4I_0(R)^2}\rho^2 - \frac{\pi(a_1^2 + b_1^2)}{2I_1(R)^2}\rho^2 + o(\rho^2). && (130)
\end{aligned}
$$

In the previous calculation, we have used expansion (27). Finally, thanks to (117) we obtain

$$
\begin{aligned}
\int_{\Gamma_R} u_0^R \partial_n (y_\rho - y_0) d\sigma &= \int_{\Gamma_R} u_0^R (g_\rho - g_0) d\sigma && (131) \\
&= -\pi\left(\frac{a_0(R) h_0^a(R) - \tilde{a}_0(0) a_0(R)}{4I_0(R)} + \frac{a_1 h_1^a(R)}{2I_1(R)}\right)\rho^2 && (132) \\
&\quad -\pi\left(\frac{b_1 h_1^b(R)}{2I_1(R)}\right)\rho^2 + o(\rho^2). && (133)
\end{aligned}
$$

Finally, with the previous expansions and noticing that $\tilde{a}_0(0) = 2f(0)$, we obtain the following result.

Theorem 5. *The energy $E(\Omega_\rho, u_\rho)$ admits the following expansion*

$$\begin{aligned}
E(\Omega_\rho, u_\rho) &= E(\Omega, u) - \left[\frac{a_0(R)^2}{8I_0(R)^2} + \frac{h_0^a(R)^2}{8} + \frac{a_0(R)h_0^a(R)}{4I_0(R)}\right]\pi\rho^2 \\
&\quad - \left[\frac{a_1(R)^2 + b_1(R)^2}{4I_1(R)^2}\right]\pi\rho^2 \\
&\quad - \left[\frac{a_1(R)h_1^a(R)}{2I_1(R)} + \frac{b_1(R)h_1^b(R)}{2I_1(R)}\right]\pi\rho^2 \\
&\quad - \left[\frac{h_1^a(R)^2}{4} + \frac{h_1^b(R)^2}{4}\right]\pi\rho^2 \\
&\quad + \left[\frac{f(0)a_0(R)}{2I_0(R)} + \frac{f(0)h_0^a(R)}{2}\right]\pi\rho^2 + o(\rho^2).
\end{aligned} \tag{134}$$

The coefficients a_0, a_1 and b_1 are given by

$$a_0(R) = \frac{1}{\pi}\int_0^{2\pi} u(R,\phi)\, d\phi,$$

$$a_1(R) = \frac{1}{\pi}\int_0^{2\pi} u(R,\phi)\sin\phi\, d\phi,$$

$$b_1(R) = \frac{1}{\pi}\int_0^{2\pi} u(R,\phi)\cos\phi\, d\phi.$$

The functions $h_i^a(R)$, $i = 1, 2$ are defined in (105).

The quantities between brackets in (134) do not depend on R, and we can easily show that

$$u(0) = \frac{a_0(R)}{2I_0(R)} + \frac{h_0^a(R)}{2}$$

and therefore

$$\frac{u(0)^2}{2} = \frac{a_0(R)^2}{8I_0(R)^2} + \frac{h_0^a(R)^2}{8} + \frac{a_0(R)h_0^a(R)}{4I_0(R)}.$$

Moreover,

$$|\nabla u(0)|^2 = \frac{a_1(R)^2 + b_1(R)^2}{4I_1(R)^2} + \frac{a_1(R)h_1^a(R)}{2I_1(R)} + \frac{b_1(R)h_1^b(R)}{2I_1(R)} + \frac{h_1^a(R)^2}{4} + \frac{h_1^b(R)^2}{4}.$$

We then deduce a different expression of the previous asymptotic expansion, which actually leads to the usual expression for the topological derivative

$$E(\Omega_\rho, u_\rho) = E(\Omega, u) + \left[-\frac{u(0)^2}{2} - |\nabla u(0)|^2 + f(0)u(0)\right]\pi\rho^2 + o(\rho^2).$$

Let us mention that, for $i = 1, 2$

$$\lim_{R\to 0} h_i^a(R) = 0, \quad \lim_{R\to 0} h_1^b(R) = 0,$$

and therefore, formula (134) gives an approximation of the topological derivative which can be calculated on the curve Γ_R, which can be interesting from a numerical point of view. In particular, it is possible to compute $a_1(R)$ and $b_1(R)$ without computing the gradient of solution u.

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